MATH 245 F17, Exam 1 Solutions

1. Carefully define the following terms: $\leq$ (for integers, as defined in Chapter 1), factorial, Associativity theorem (for propositions), Distributivity theorem (for propositions).

Let a, b be integers. We say that $a \leq b$ if $b - a \in \mathbb{N}_0$. The factorial is a function from $\mathbb{N}_0$ to $\mathbb{Z}$ (or $\mathbb{N}$), denoted by !, defined by: $0! = 1$ and $n! = (n-1)! \cdot n$ (for $n \geq 1$). The Associativity theorem says: Let p, q, r be propositions. Then $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$ and also $(p \vee q) \vee r \equiv p \vee (q \vee r)$. The Distributivity theorem says: Let p, q, r be propositions. Then $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$ and also $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$.

2. Carefully define the following terms: Addition semantic theorem, Contrapositive Proof theorem, Direct Proof, converse.

The Addition semantic theorem states that for any propositions p, q, we have $p \vdash p \vee q$. The Contrapositive Proof theorem states that for any propositions p, q, if $(\neg q) \vdash (\neg p)$ is valid, then $p \to q$ is T The Direct Proof theorem states that for any propositions p, q, if $p \vdash q$ is valid, then $p \to q$ is T. The converse of conditional proposition $p \to q$ is $q \to p$.

3. Let a, b be odd. Prove that $4a - 3b$ is odd.

Because a is odd, there is integer c with $a = 2c + 1$. Because b is odd there is integer d with $b = 2d + 1$. Now, $4a - 3b = 4(2c+1) - 3(2d+1) = 8c + 4 - 6d - 3 = 2(4c - 3d) + 1$. Because $4c - 3d$ is an integer, $4a - 3b$ is odd.

4. Suppose that $a|b$. Prove that $a|(4a - 3b)$.

Because $a|b$, there is integer c with $b = ca$. Now, $4a - 3b = 4a - 3(ca) = a(4 - 3c)$. Because $4 - 3c$ is an integer, $a|(4a - 3b)$.

5. Simplify $\neg((p \to q) \vee (p \to r))$ to use only $\neg, \vee, \wedge$, and to have only basic propositions negated.

Applying De Morgan's law, we get $(\neg(p \to q)) \wedge (\neg(p \to r))$. Applying a theorem from the book (2.16), we get $(p \wedge (\neg q)) \wedge (p \wedge (\neg r))$. Applying associativity and commutativity of $\wedge$ several times, we get $(p \wedge p) \wedge (\neg q) \wedge (\neg r)$. Applying a theorem from the book (2.7), we get $p \wedge (\neg q) \wedge (\neg r)$.

6. Without truth tables, prove the Constructive Dilemma theorem, which states: Let p, q, r, s be propositions. $p \to q, r \to s, p \vee r \vdash q \vee s$.

Because $p \vee r$ is T (by hypothesis), we have two cases: p is T or r is T. If p is T, we apply modus ponens to $p \to q$ to conclude q. We then apply addition to get $q \vee s$. If instead r is T, we apply modus ponens to $r \to s$ to conclude s. We apply addition to get $q \vee s$. In both cases $q \vee s$ is T.

7. State the Conditional Interpretation theorem, and prove it using truth tables.

The CI theorem states:
Let p, q be propositions. Then $p \to q \equiv q \vee (\neg p)$.
Proof: The third and fifth columns in the truth table at right, as shown, agree. Hence $p \to q \equiv q \vee (\neg p)$.

p	q	$p \to q$	$\neg p$	$q \vee (\neg p)$
T	T	T	F	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

8. Let $x \in \mathbb{R}$. Suppose that $\lfloor x \rfloor = \lceil x \rceil$. Prove that $x \in \mathbb{Z}$.

First, $\lfloor x \rfloor \leq x$ by definition of floor. Second, $x \leq \lceil x \rceil$ by definition of ceiling. But since $\lfloor x \rfloor = \lceil x \rceil$, in fact $x \leq \lfloor x \rfloor$. Combining with the first fact, $x = \lfloor x \rfloor$. Since $\lfloor x \rfloor$ is an integer, so is x.

9. Prove or disprove: For arbitrary propositions p, q, $(p \downarrow q) \to (p \uparrow q)$ is a tautology.

Since the fifth column in the truth table at right, as shown, is all T, the proposition $(p \downarrow q) \to (p \uparrow q)$ is indeed a tautology.

p	q	$p \downarrow q$	$p \uparrow q$	$(p \downarrow q) \to (p \uparrow q)$
T	T	F	F	T
T	F	F	T	T
F	T	F	T	T
F	F	T	T	T

10. Prove or disprove: For arbitrary $x \in \mathbb{R}$, if x is irrational then $2x - 1$ is irrational.

The statement is true, we provide a contrapositive proof. Suppose that $2x - 1$ is rational. Then there are integers a, b, with b nonzero, such that $2x - 1 = \frac{a}{b}$. We have $2x = \frac{a}{b} + 1 = \frac{a+b}{b}$, and $x = \frac{a+b}{2b}$. Now, $a + b, 2b$ are integers, and $2b$ is nonzero, so x is rational.